Journal of Applied Mechanics and Technical Physics, Vol. 48, No. 5, pp. 737–742, 2007

ANALYSIS OF THE DERIVATIVE OF THE ENERGY FUNCTIONAL WITH RESPECT TO THE LENGTH OF A CURVILINEAR CRACK IN AN ELASTIC BODY WITH A POSSIBLE CRACK-EDGE CONTACT

E. V. Vtorushin

UDC 539.375

A homogeneous two-dimensional body with a crack of variable length is considered. At the crack edges, conditions are formulated in the form of inequalities that describe mutual nonpenetration of the edges. The derivative of the elastic-energy functional with respect to the length of the curvilinear crack is analyzed. It is shown that the derivative is independent of the crack path, provided that the curve along which the crack propagates is reasonably smooth.

Key words: *derivative of the energy functional, nonpenetration, variational inequality, Griffith criterion.*

Introduction. The present paper addresses some mathematical problems of the crack theory [1], in particular, crack propagation in elastic bodies. In this theory, the Griffith criterion plays an important role, according to which the crack starts growing if the derivative of the elastic-energy functional with respect to the crack length reaches the critical value $2\gamma_0$. This quantity is a material characteristic of the medium. The derivative of the energy functional is evaluated along the crack propagation path. The dependence of the derivative on the path is studied, which is important for understanding the Griffith criterion.

The equilibrium problem of a solid body is studied, based on the two-dimensional theory of elasticity [1]. The body contains a crack. At the crack edges, the nonpenetration conditions are formulated in the form of a system of equalities and inequalities. The external boundary is subjected to the Dirichlet homogeneous conditions.

At present, there exists a large body of literature where parameter-dependent solutions of elliptic equations are studied for various perturbations of the domains. The case of smooth domains was considered in [2]. Results concerning the differentiation of energy functionals for linear boundary-value problems in nonsmooth domains were given in [3, 4].

For nonlinear elliptic problems with boundary conditions formulated in the form of inequalities, the derivative of the energy functional was first obtained in [5]. The method for obtaining the derivative described in [5] allows one to avoid calculating boundary conditions for the material derivative of the solution, which is generally determined nonuniquely. Later, similar formulas were derived for derivatives in various problems of the elasticity theory [6–9] with the use of variational formulations [10]. It was assumed thereby that the cracks were rectilinear; otherwise, additional conditions were imposed on the domain perturbation, which transformed the set of admissible displacements of points of the body in the unperturbed problem into that in the perturbed problem.

With the help of the formulas obtained, invariant integrals of the Cherepanov–Rice [1] type were derived. This integral determines the energy-release rate for quasi-static crack growth and is used in fracture mechanics to model the crack growth.

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; vtorushin@ngs.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 48, No. 5, pp. 138–145, September–October, 2007. Original article submitted November 1, 2005; revision submitted September 19, 2006.

Brokate and Khludnev [11] showed that the derivative of the energy functional along the curvilinear crack path in a two-dimensional elastic isotropic body does not depend on the path shape if the curve describing this path is reasonably smooth. In this case, linear boundary conditions were specified at the crack.

Rudoy [12] obtained a formula for the derivative of the energy functional along a curvilinear path in a two-dimensional problem of the theory of elasticity. At the crack edges, nonlinear boundary conditions were formulated in the form of equalities and inequalities. The main difficulty was to construct one-to-one mapping between the sets of admissible displacements in the perturbed and unperturbed problems.

The last findings concerning the study of curvilinear cracks propagating in elastic bodies under nonpenetration constraints are given [13].

Perturbation of the Equilibrium Problem of an Elastic Body with a Crack under Conditions of a Possible Edge Contact. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary Γ. We denote the set $[-1,0]\times\{0\}$ by Σ and consider the plot Γ^δ of the function $x_2=\psi(x_1)$ for $x_1 \in (0,\delta)$ and $\psi(0)=\psi'(0)=0$, where $(x_1,x_2)\in\Omega,\ \delta\geq 0$. We set $\Gamma_\delta=\Sigma\cup\Gamma^\delta$ assuming that $\bar{\Gamma}_\delta\subset\Omega$ for a small varied parameter δ. In a particular case with $\delta=0$, we obtain $\Gamma_0=\Sigma$. The function ψ is assumed to be reasonably smooth. The question of which specific class this function belongs to is discussed below. We consider the domain Ω_δ with a cut (crack) $\bar{\Gamma}_\delta$, i.e., $\Omega_\delta=\Omega\setminus\bar{\Gamma}_\delta$.

Defining the unit normal vector $\boldsymbol{\nu}$ on Γ_δ as

$$\boldsymbol{\nu}=\begin{cases} (0,1) & \text{on } \Sigma,\\ \Big(-\psi_{x_1}/\sqrt{1+\psi_{x_1}^2},1/\sqrt{1+\psi_{x_1}^2}\Big) & \text{on } \Gamma^\delta,\end{cases}$$

one can identify the positive and negative edges $\Gamma_\delta^\pm$ of the crack with respect to the vector $\boldsymbol{\nu}$.

In the domain Ω_0, we consider an equilibrium problem of an elastic body with a crack Γ_0 under nonpenetration conditions. If the right side $\boldsymbol{f}=(f_1,f_2)\in[C^1(\bar{\Omega}_0)]^2$ is known, the displacement vector $\boldsymbol{u}=(u_1,u_2)$ satisfies the equilibrium equations subject to boundary conditions in the form of equalities and inequalities:

$$-\sigma_{ij,j}=f_i,\qquad i,j=1,2\qquad \text{in}\quad \Omega_0; \tag{1}$$

$$\sigma_{ij}=c_{ijkl}\varepsilon_{kl}(\boldsymbol{u}),\qquad i,j,k,l=1,2\qquad \text{in}\quad \Omega_0; \tag{2}$$

$$\boldsymbol{u}=0\qquad \text{on}\quad \Gamma; \tag{3}$$

$$[\boldsymbol{u}]\cdot\boldsymbol{\nu}\geq 0,\quad \sigma_\nu\leq 0,\quad [\sigma_\nu]=0,\quad \sigma_\tau=0,\quad \sigma_\nu[\boldsymbol{u}]\cdot\boldsymbol{\nu}=0\qquad \text{on}\quad \Gamma_0^\pm. \tag{4}$$

Here $\varepsilon_{kl}(\boldsymbol{u})=(u_{k,l}+u_{l,k})/2$ are the strain-tensor components, $u_{k,l}=\partial u_k/\partial x_l$, $\boldsymbol{x}=(x_1,x_2)\in\Omega_0$, $\sigma_{ij}=\sigma_{ij}(\boldsymbol{u})$ are the stress-tensor components, and c_{ijkl} are the components of the positively defined tensor of elastic coefficients ($c_{ijkl}=c_{jikl}=c_{klij}$, $c_{ijkl}\xi_{kl}\xi_{ij}\geq c_0\xi_{ij}\xi_{ij}$, $c_0>0$, and $\xi_{ij}=\xi_{ji}$). To simplify calculations, we assume that c_{ijkl} are constants. Finally, we use the decomposition

$$\{\sigma_{ij}\}=\sigma_\nu\boldsymbol{\nu}+\sigma_\tau,\qquad \sigma_\nu=\sigma_{ij}\nu_j\nu_i,\qquad i,j=1,2$$

and the following notation: $[v]=v^+-v^-$ is the discontinuity of the function v on Γ_0 and $v^\pm$ are the values of the function v on $\Gamma_0^\pm$.

For the small parameter δ, we consider perturbation of problem (1)–(4). Let the unknown displacement vector $\boldsymbol{u}$ satisfy the following system of equations with boundary conditions:

$$\begin{gathered}-\sigma^\delta_{ij,j}=f_i,\qquad i,j=1,2\qquad \text{in}\quad \Omega_\delta,\\ \sigma^\delta_{ij}=c_{ijkl}\varepsilon_{kl}(\boldsymbol{u}^\delta),\qquad i,j,k,l=1,2\qquad \text{in}\quad \Omega_\delta,\\ \boldsymbol{u}^\delta=0\qquad \text{on}\quad \Gamma,\\ [\boldsymbol{u}^\delta]\cdot\boldsymbol{\nu}\geq 0,\quad \sigma^\delta_\nu\leq 0,\quad [\sigma^\delta_\nu]=0,\quad \sigma^\delta_\tau=0,\quad \sigma^\delta_\nu[\boldsymbol{u}^\delta]\cdot\boldsymbol{\nu}=0\qquad \text{on}\quad \Gamma_\delta^\pm.\end{gathered} \tag{5}$$

Problem (5) admits variational formulation, and its solution corresponds to the minimum of the elastic-energy functional

$$\Pi_\psi(\boldsymbol{v};\delta) = \frac{1}{2}\int\limits_{\Omega_\delta} \sigma_{ij}(\boldsymbol{v})\varepsilon_{ij}(\boldsymbol{v}) - \int\limits_{\Omega_\delta} \boldsymbol{f}\cdot\boldsymbol{v}$$

on the convex set

$$K_\delta = \{\boldsymbol{v} \in H_0^1(\Omega_\delta)\colon [\boldsymbol{v}]\cdot\boldsymbol{\nu} \geq 0 \quad \text{almost everywhere on } \Gamma_\delta\},$$

where

$$H_0^1(\Omega_\delta) = \{\boldsymbol{v} = (v_1, v_2)\colon v_i \in H^1(\Omega_\delta),\ i = 1,2;\ \boldsymbol{v} = 0 \text{ on } \Gamma\}.$$

The minimization problem can be formulated as follows: find an element $\boldsymbol{u}^\delta \in K_\delta$ such that

$$\Pi_\psi(\boldsymbol{u}^\delta;\delta) = \min_{\boldsymbol{v}\in K_\delta} \Pi_\psi(\boldsymbol{v};\delta).$$

The element $\boldsymbol{u}^\delta$ is the solution of the variational inequality. It is required to find an element $\boldsymbol{u}^\delta \in K_\delta$ such that

$$\int\limits_{\Omega_\delta} \sigma_{ij}(\boldsymbol{u}^\delta)\varepsilon_{ij}(\boldsymbol{v}-\boldsymbol{u}^\delta) \geq \int\limits_{\Omega_\delta} f(\boldsymbol{v}-\boldsymbol{u}^\delta), \qquad \boldsymbol{v}\in K_\delta. \tag{6}$$

For $\delta = 0$, the solution $\boldsymbol{u}^0$ of the variational problem (6) coincides with the solution $\boldsymbol{u}$ of the unperturbed problem (1)–(4). It should be noted that problem (6) has a unique solution $\boldsymbol{u}^\delta$ for all values of δ.

We determine the derivative of the energy functional $\Pi_\psi(\boldsymbol{u}^\delta;\delta)$ with respect to the perturbation parameter δ. For this purpose, following [7, 8], we construct a mapping of the perturbed domain Ω_δ onto the original domain Ω_0. We consider the function $\theta \in C_0^\infty(\Omega)$ with a carrier in a reasonably small neighborhood of $(0,0)$, such that $\theta \equiv 1$ in a still smaller neighborhood of $(0,0)$ of radius r_δ. For small $\delta < r_\delta$, we introduce a transformation of independent variables $y = y(x,\delta)$, where $y \in \Omega_0$ and $x \in \Omega_\delta$:

$$y_1 = x_1 - \delta\theta(x_1,x_2), \qquad y_2 = x_2 + \psi(x_1 - \delta\theta(x_1,x_2)) - \psi(x_1). \tag{7}$$

The Jacobian of transformation (7) is given by

$$J_\delta = \Big|\frac{\partial y(x,\delta)}{\partial x}\Big| = 1 - \delta\,\frac{\partial\theta}{\partial\boldsymbol{\tau}}, \qquad \frac{\partial}{\partial\boldsymbol{\tau}} \equiv \frac{\partial}{\partial x_1} + \psi'(x_1)\,\frac{\partial}{\partial x_2}.$$

It is obvious that $J_\delta > 0$ for small δ. Thus, transformation (7) maps the perturbed domain Ω_δ onto the unperturbed domain Ω_0 in a one-to-one manner. In particular, the point $(\delta, \psi(\delta))$ is mapped onto the point $(0,0)$. It should be noted that the argument of the function $\psi(x_1)$ can be negative. In this case, to keep all formulas valid, we set $\psi(x_1) = 0$ for all $x_1 < 0$. Then, the domain Ω_δ ($\delta < 0$) is well defined. The function ψ is assumed to be reasonably smooth within the interval $(-1, \delta)$.

Rudoy [12] found that the derivative $\Pi'_\psi(0) = (d/d\delta)\Pi_\psi(\boldsymbol{u}^\delta;\delta)\Big|_{\delta=0}$ of the energy functional with respect to the length δ of the crack projection Γ_δ onto the x_1 axis is given by

$$\Pi'_\psi(0) = \frac{1}{2}\int\limits_{\Omega_0} \frac{\partial\theta}{\partial\boldsymbol{\tau}}\,\sigma_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{u}) - \int\limits_{\Omega_0} \sigma_{ij}(\boldsymbol{u})\boldsymbol{E}_{ij}(\theta;\boldsymbol{u})$$

$$- \int\limits_{\Omega_0} \frac{\partial}{\partial\boldsymbol{\tau}}\,(\theta f_i)u_i + \int\limits_{\Omega_0} \sigma_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{Q}) - \int\limits_{\Omega_0} \boldsymbol{f}\cdot\boldsymbol{Q}, \tag{8}$$

where

$$\boldsymbol{E}_{ij}(\theta;\boldsymbol{u}) = \frac{1}{2}\Big(\theta_{,j}\,\frac{\partial u_i}{\partial x_1} + \theta_{,i}\,\frac{\partial u_j}{\partial x_1}\Big) + (\theta\psi')_{,j}\,\frac{\partial u_i}{\partial x_2} + (\theta\psi')_{,i}\,\frac{\partial u_j}{\partial x_2}, \qquad \boldsymbol{Q} = (0, \psi'' u_1).$$

Formula (8) is used in applications. In particular, the following approximation of the energy functional was obtained:

$$\Pi_\psi(\boldsymbol{u}^\delta;\delta) = \Pi_\psi(\boldsymbol{u};0) + \delta\Pi'_\psi(0) + o(\delta).$$

Rudoy demonstrated [12] that the derivative of the energy functional (8) is independent of the function θ.

Independence of the Derivative $\boldsymbol{\Pi'_\psi(0)}$ of the Function $\boldsymbol{\psi}$. Below, we show that the derivative $\Pi'_\psi(0)$ is independent of the function ψ if this function is reasonably smooth.

We introduce a space of functions belonging to $H^4(0,1)$ such that

$$H^{4,0}(0,1) = \{\xi \in H^4(0,1)\colon \xi(0) = \xi'(0) = \xi''(0) = \xi'''(0) = 0\}.$$

We further assume that $\psi \in H^{4,0}(0,1)$. Since $\psi = 0$ for $x_1 < 0$, the inclusion $\psi \in H^4(-1,1)$ holds.

Let B be a sphere of small radius such that the carrier of the function θ lies inside B ($B_\Sigma = B \setminus \Sigma$). The derivative (8) can be written as

$$\Pi'_\psi(0) = \int\limits_{B_\Sigma} d + \int\limits_{B_\Sigma} q\psi' + \int\limits_{B_\Sigma} r\psi'' + \int\limits_{B_\Sigma} s\psi''', \tag{9}$$

where $d = d(D^\alpha \boldsymbol{u}, D^\alpha\theta, D^\alpha \boldsymbol{f})$, $q = q(D^\alpha \boldsymbol{u}, D^\alpha\theta, D^\alpha \boldsymbol{f})$, $r = r(D^\alpha \boldsymbol{u}, \theta, \boldsymbol{f})$, and $s = s(D^\alpha \boldsymbol{u})$ are known functions; $|\alpha| \le 1$; $\alpha = (\alpha_1, \alpha_2)$. We note that the smoothness properties of the functions $\boldsymbol{u} \in H^1_0(\Omega_0)$ and $\psi \in H^{4,0}(0,1)$ imply that all integrals in (9) are bounded. Indeed, the terms on the right side of equality (9) have the form $u_{,i}u_{,j}\psi'$, $u_{,i}u_{,j}\psi''$, and $u_{,i}u_{,j}\psi'''$. According to the nested theorem for the Sobolev space, we have $\psi', \psi'', \psi''' \in L^\infty(-1,1)$. Moreover, $u_{,i}u_{,j} \in L^1(B_\Sigma)$ and, hence, the statement is valid.

We introduce the notation $B^+_\Sigma = \{(x_1,x_2) \in B_\Sigma\colon\ x_1 > 0\}$. Then, the formula for derivative (9) can be written as

$$\Pi'_\psi(0) = \int\limits_{B_\Sigma} d + \int\limits_{B^+_\Sigma} q\psi' + \int\limits_{B^+_\Sigma} r\psi'' + \int\limits_{B^+_\Sigma} s\psi'''. \tag{10}$$

Here the equality $\psi(x_1) = 0$ valid for all $-1 < x_1 < 0$ was taken into account.

Formula (10) explicitly expresses the dependence $\Pi'_\psi(0)$ on the function ψ. Below, we prove the independence of the derivative $\Pi'_\psi(0)$ of ψ if $\psi \in H^{4,0}(0,1)$. In essence, it is shown that the sum of the integrals over B^+_Σ in (10) vanishes. This implies, in particular, that the Griffith criterion is indifferent to any crack growth along the trajectory determined by the function $\psi \in H^{4,0}(0,1)$.

We introduce the notation

$$N(\psi) = \int\limits_{B^+_\Sigma} q\psi' + \int\limits_{B^+_\Sigma} r\psi'' + \int\limits_{B^+_\Sigma} s\psi'''$$

and prove the following theorem.

Theorem 1. *The following identity is valid:*

$$N(\psi) = 0, \qquad \psi \in H^{4,0}(0,1).$$

Proof. We consider a function φ of the form

$$\varphi(x_1) = kx_1^n, \qquad x_1 \in (0,1), \quad k \in \mathbb{R}, \quad n = 3,4,\dots\ . \tag{11}$$

In this case, we have

$$\varphi \in H^{4,0}(0,1), \qquad n = 3,4,\dots\ .$$

We introduce additional notation. Let

$$a = \int\limits_{B_\Sigma} d, \qquad b_n = n \int\limits_{B^+_\Sigma} \Big(qx_1^{n-1} + (n-1)rx_1^{n-2} + (n^2-3n+2)rx_1^{n-3}\Big), \quad n = 3,4,\dots\ .$$

Substituting the functions φ of the form (11) into formula (10), we evaluate the derivative $\Pi'_\psi(0)$. As a result, we obtain the relations

$$\Pi'_\varphi(0) = a + kb_n, \qquad n = 3,4,\dots\ . \tag{12}$$

The derivative of the energy functional has a constant sign. Indeed, for any positive δ, the following relation holds:

$$\frac{\Pi_\psi(\boldsymbol{u}^\delta;\delta) - \Pi_\psi(\boldsymbol{u}^0;0)}{\delta} \le \frac{\Pi_\psi(\boldsymbol{u}^0;\delta) - \Pi_\psi(\boldsymbol{u}^0;0)}{\delta} = 0.$$

In other words, the inequality $\Pi'_\psi(0) \le 0$ is satisfied for all $\psi \in H^{4,0}(0,1)$; in particular, the following inequality should hold:

$$\Pi'_\varphi(0) \le 0. \tag{13}$$

As the value of k is chosen arbitrarily, it follows from Eqs. (12) and (13) that

$$b_n = 0, \qquad n = 3, 4, \ldots \;.$$

Then, $N(\varphi) = 0$ for all functions φ of the form (11). Consequently, for all polynomials of the form

$$\varphi(x_1) = \beta_3 x_1^3 + \beta_4 x_1^4 + \ldots + \beta_n x_1^n \qquad (n = 3, 4, \ldots) \tag{14}$$

we obtain

$$N(\varphi) = 0. \tag{15}$$

We show that polynomials of the form of (14) are dense in the space $H^{4,0}(0,1)$. Let $\psi \in H^{4,0}(0,1)$ be a certain fixed function. It is known that the functions belonging to the space $C^4[0,1]$ and vanishing in the neighborhood of $x_1 = 0$ are dense in the space $H^{4,0}(0,1)$. At the same time, for any function $\chi \in C^4[0,1]$, there exists a sequence of the Bernstein polynomials converging to $\chi \in C^4[0,1]$ in the space $C^4[0,1]$. The Bernstein polynomials are determined by the formula [14]

$$\chi(x_1) = \sum_{k=0}^{n} \chi\Big(\frac{k}{n}\Big) C_n^k x_1^k (1 - x_1)^{n-k}, \tag{16}$$

where $C_n^k = n!/((n-k)!k!)$. We take an arbitrary value of $\lambda > 0$. There exists a function $\chi \in C^4[0,1]$ equal to zero in the neighborhood of $x_1 = 0$ such that

$$\|\chi - \psi\|_{H^{4,0}(0,1)} < \lambda. \tag{17}$$

For the function χ, we construct a sequence of the Bernstein polynomials χ_n of the form (16). For sufficiently large n, we obtain

$$\|\chi - \chi_n\|_{C^4[0,1]} < \lambda. \tag{18}$$

Note that $\chi(1/n) = 0$ and $\chi(2/n) = 0$ for large n. By virtue of (16), this means that the Bernstein polynomials χ_n become

$$\chi_n(x_1) = \beta_3 x_1^3 + \beta_4 x_1^4 + \ldots + \beta_n x_1^n.$$

According to Eqs. (17) and (18), we have

$$\|\chi_n - \psi\|_{H^{4,0}(0,1)} < 2\lambda.$$

Thus, the density of polynomials of the form (14) in the space $H^{4,0}(0,1)$ has been proven.

Since equality (15) is valid for all polynomials of the form of (14), it is also valid for all $\psi \in H^{4,0}(0,1)$. Thus, Theorem 1 has been proven.

It follows from Theorem 1 that the derivative of the energy functional can be written as

$$\Pi'_\psi(0) = a, \qquad \psi \in H^{4,0}(0,1).$$

The right side of the equality a does not depend on the function ψ. Calculating the value of a, we obtain

$$\Pi'_\psi(0) = \int\limits_{\Omega_0} \frac{1}{2}\,\sigma_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{u})\theta_{,1} - \sigma_{ij}(\boldsymbol{u})u_{i,1}\theta_{,j} - \int\limits_{\Omega_0} u_i(f_i\theta)_{,1}. \tag{19}$$

Independence of the Derivative $\Pi'_\psi(0)$ of the Function θ. In formula (19), integration can be performed only with respect to the carrier θ. The solution $\boldsymbol{u}$ of problem (1)–(4) is known to belong to the Sobolev space H^2 except for the neighborhood of the crack tip. We recall that $\theta = 1$ in the neighborhood of the point $(0,0)$. Let B^0 be a certain sphere centered at the point $(0,0)$, such that $B^0 \subset \{x \in \Omega\colon\ \theta(x) = 1\}$. We introduce the set $B^0_\Sigma = B^0 \setminus \Sigma$ and integrate (19) by parts. Then, for $\varepsilon_{ij} = \varepsilon_{ij}(\boldsymbol{u})$ and $\sigma_{ij} = \sigma_{ij}(\boldsymbol{u})$, we obtain

$$\Pi'_\psi(0) = -\int\limits_{\Omega_0\setminus B^0_\Sigma} \Big(\Big(\frac{1}{2}\,\sigma_{ij}\varepsilon_{ij}\Big)_{,1} - (\sigma_{ij}u_{i,1})_{,j} - u_{i,1}f_i\Big)\theta + \int\limits_{\Sigma\setminus B^0} [\sigma_{ij}\boldsymbol{\nu}_j u_{i,1}]\theta + \int\limits_{\partial B^0} \Big(\sigma_{ij}u_{1,i}n_j - \frac{1}{2}\,\sigma_{ij}\varepsilon_{ij}n_1\Big) + \int\limits_{B^0_\Sigma} u_{i,1}f_i. \tag{20}$$

Here $\boldsymbol{n} = (n_1, n_2)$ is the outward normal vector to the boundary ∂B^0 of the sphere B^0. Since the quantities $\varepsilon_{ij}(\boldsymbol{u})$, $\sigma_{ij}(\boldsymbol{u})$, and f_i are related by the equilibrium equations (1) and Hooke's law (2), the integral over $\Omega_0 \setminus B^0_\Sigma$ vanishes.

The integral over $\Sigma \setminus B^0$ can be written as

$$\int\limits_{\Sigma\setminus B^0} [\sigma_{ij}\boldsymbol{\nu}_j u_{i,1}]\theta = \int\limits_{\Sigma\setminus B^0} [\sigma_{12}u_{1,1}]\theta + \int\limits_{\Sigma\setminus B^0} [\sigma_{22}u_{2,1}]\theta. \tag{21}$$

Rudoy [12] analyzed the right side of (21) using the results of [15] and found that satisfaction of conditions (4) ensures satisfaction of the equality

$$[\sigma_{22}u_{2,1}] = 0 \tag{22}$$

almost everywhere. It follows from Eqs. (4) and (22) that the integral over $\Sigma \setminus B^0$ in (20) vanishes. As a result, we write the formula for the energy derivative that does not contain the function θ:

$$\Pi'_\psi(0) = \int\limits_{\partial B^0} \Big(\sigma_{ij}u_{1,i}n_j - \frac{1}{2}\,\sigma_{ij}\varepsilon_{ij}n_1\Big) + \int\limits_{B^0_\Sigma} u_{i,1}f_i. \tag{23}$$

The right side of Eq. (23) depends neither on ψ nor on the choice of the sphere B^0. This implies that the right side of Eq. (23) remains unchanged as the sphere radius decreases. If the components of the mass-force vector f_i vanish inside the sphere B^0, the right side of Eq. (23) is the Cherepanov–Rice integral, which is often used in applications.

Remark 1. The results obtained can be generalized to the following case within the two-dimensional theory of elasticity. Let the crack be such that it coincides with a straight line in a certain neighborhood of the tip. It follows that the derivative of the energy functional along a possible trajectory of crack propagation is independent of the trajectory shape if the direction of possible propagation of the crack coincides with the direction of the tangent at the crack tip.

This work was supported by the Russian Foundation for Basic Research (Grant No. 06-01-00209).

REFERENCES

1. N. F. Morozov, *Mathematical Problems of the Cracks Theory* [in Russian], Nauka, Moscow (1984).
2. J. Sokolowski and J. P. Zolesio, *Introduction to Shape Optimization. Shape Sensitivity Analysis*, Springer, Berlin (1992). (Springer Series in Computational Mathematics; Vol. 10.)
3. K. Othsuka, "Mathematics of brittle fracture," in: *Theoretical Studies on Fracture Mechanics in Japan*, Hiroshima-Denki Inst. of Technol., Hiroshima (1997), pp. 99–172.
4. V. G. Maz'ya and S. A. Nazarov, "Asymptotic behavior of energy integrals under small perturbations of the boundary near the corner and conic points," *Tr. Mosk. Mat. Obshch.*, **50**, 79–129 (1987).
5. A. M. Khludnev and J. Sokolowski, "The Griffith formula and the Rice–Cherepanov integral for crack problems with unilateral conditions in nonsmooth domains," *Europ. J. Appl. Math.*, **10**, No. 4, 379–394 (1999).
6. A. M. Khludnev, K. Ohtsuka, and J. Sokolowski, "On derivative of energy functional for elastic bodies with cracks and unilateral conditions," *Quart. Appl. Math.*, **60**, 99–109 (2002).
7. E. M. Rudoy, "Griffith formula for a cracked plate," *Sib. Zh. Indust. Mat.*, **60**, 155–161 (2002).
8. E. M. Rudoy, "Asymptotic behavior of the energy integral for perturbation of the boundary," in: *Dynamics of Continuous Media* (collected scientific papers) [in Russian], No. 116, Inst. of Hydrodynamics, Sib. Div., Russian Acad. of Sci., Novosibirsk (2000), pp. 97–103.
9. V. A. Kovtunenko, "Shape sensitivity of curvilinear cracks on interface to non-linear perturbations," *Z. Angew. Math. Phys.*, **54**, No. 3, 410–423 (2003).
10. A. Khludnev and V. Kovtunenko, *Analysis of Cracks in Solids*, WIT-Press, Boston (2000).
11. M. Brokate and A. Khludnev, "On crack propagation shapes in elastic bodies," *Z. Angew. Math. Phys.*, **55**, No. 2, 318–329 (2004).
12. E. M. Rudoy, "Differentiation of energy functionals in two-dimensional elasticity theory for solids with curvilinear cracks," *J. Appl. Mech. Techn. Phys.*, **45**, No. 6, 843–852 (2004).
13. V. A. Kovtunenko, "Primal-dual methods of shape sensitivity analysis for curvilinear cracks with nonpenetration," *IMA J. Appl. Math.*, **71**, No. 5, 635–657 (2006).
14. S. N. Bernstein, *Collected Papers* [in Russian], Vol. 1, Izd. Akad. Nauk SSSR, Moscow (1952).
15. D. Kinderlerer and G. Stampcchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York (1980).